CALCULUS 1 SYLLABUS

A draft of FECE-UP (Prishtina) for one subject.

- Functions, generalities– FECE-UP
- Limit and continuity– FECE-UP
- Derivation– FECE-UP
- The integral – FECE-UP

Functions, generalities	Basic notions on functions Graph of standard functions
Limit and continuity	Sequences, convergence of sequences
Derivation	Derivations rules
The integral	Indefinite integral (anti derivative)

I. Functions, generalities

1.1. Basic notions on functions

The fundamental objects in calculus are functions. We will discuss about the basic notions on functions and some examples on application of this functions. Functions arise whenever one quantity depends on another. Functions can be representing on four ways.

1. The area A of a circle depends on the radius of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a function of r. (the analytic form).

2. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

   But for each value of the time t, there is a corresponding value of P and we say that P is a function of t. (the tabular form).

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

Table 1

3. The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known. (the verbal form).

4. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t the graph provides a corresponding value of a. (the graphic form).

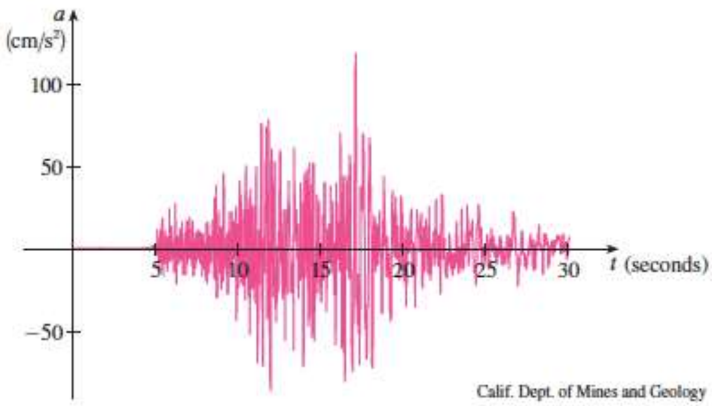


Figure 1

Each of these examples describes a rule whereby, given a number (r, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number.

Example 1.
(a) As dry air moves upward, it expands and cools. If the ground temperature is 20^0C and the temperature at a height of1 km is 10^0C, express the temperature T (in ^{0}C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
(b) Draw the graph of the function in part (a). What does the slope represent?
(c) What is the temperature at a height of 2.5 km?

Solution: Let
(a) Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y-intercept is $b = 20$.
We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

(b) The graph is sketched in Figure 2. The slope is $m = -10^0C/km$, and this represents the rate of change of temperature with respect to height.
(b) At a height of $h = 2.5\ km$, the temperature is

$$T = -10(2.5) + 20 = -5^0C$$

If there is no physical law or principle to help us formulate a model, we constract an empirical model, which is based enterely on collected data. We seek a curve that "fits" the data in the sense that it captures the basic trend of the data points.

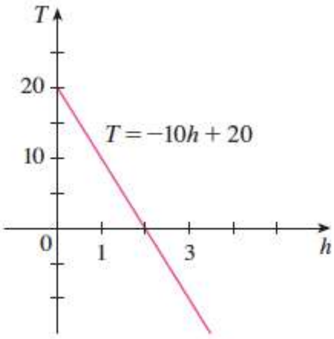


Figure 2

Example 2. Express the hypotenuse h of a right triangle with area $25m^2$ as a function of its perimeter *P*.

Solution: Let's first sort out the information by identifying the unknown quantity and the data:

Unknown: hypotenuse h

Given quantities: perimeter *P*, area $25m^2$

It helps to draw a diagram and we do so in Figure 3.

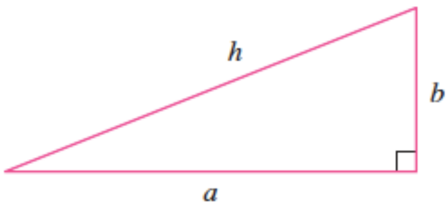


Figure 3

In order to connect the given quantities to the unknown, we introduce two extra variables *a* and *b*, which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$25 = \frac{1}{2}ab \qquad P = a + b + h$$

Since P is given, notice that we now have three equations in the three unknowns $a, b,$ and h:

$$h^2 = a^2 + b^2 \qquad (1)$$
$$25 = \frac{1}{2}ab \qquad (2)$$

$$P = a + b + h \qquad (3)$$

Alhought we have the correct number of equations, they are not easy to solve in a straight-forward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of equations1, 2, and 3. Do these expressions remind you of anything familiar?

Notice that they contain the ingredients of a familiar formula:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a+b)^2$ in two ways. From equations 1 and 2 we have

$$(a+b)^2 = (a^2+b^2) + 2ab = h^2 + 4(25)$$

From equation 3 we have:

$$(a+b)^2 = (P-h)^2 = P^2 - 2Ph + h^2$$

$$h^2 + 100 = P^2 - 2Ph + h^2$$

$$2Ph = P^2 - 100$$

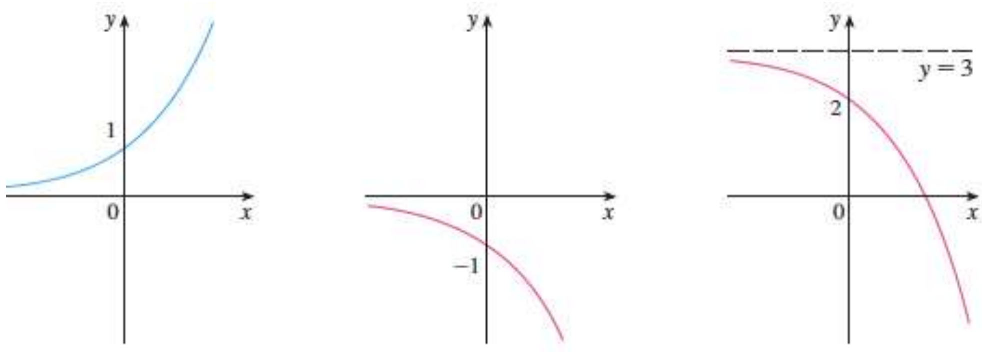
$$h = \frac{P^2 - 100}{2P}$$

This is the required expression for *h* as a function of *P*.

1.2. Graph of standard functions

Example 3. Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

Solution: First we reflect the graph of $y = 2^x$ [shown in Figure 4(a)] about the *x*-axis to get the graph $y = -2^x$ in Figure 4(b). Then we shift the graph of $y = -2^x$ upward 3 units to obtain the graph of $y = 3 - 2^x$ in Figure 4(c). The domain is $\mathbb{R}$ and the range is $(-\infty, 3)$.



(a) $y = 2^x$ (b) $y = -2^x$ (c) $y = 3 - 2^x$

Figure 4

Example 3. Sketch the graph of the function $y = ln(x - 2) - 1$.

Solution: We start with the graph of $y = lnx$ as given in Figure 5. We shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 5.)

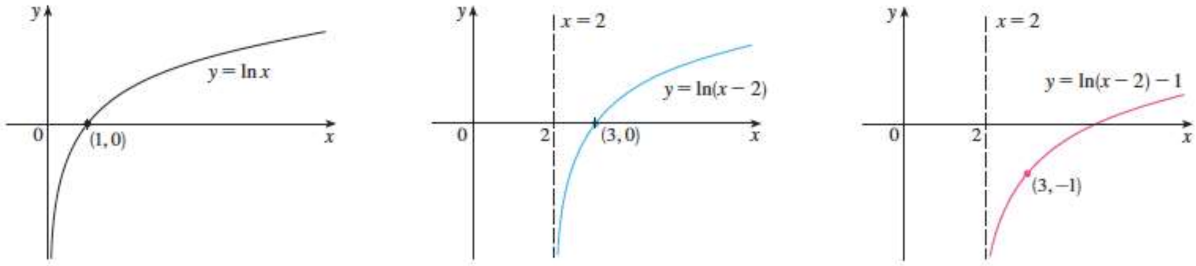


Figure 5

II. Limit and continuity
Sequences, convergence of sequences

In the fifth century BC the Greek philosopher Zeno of Elena posed four problems, now known as Zeno's paradoxes. Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise that has been given a head start. Zeno argued, as follows, that Achilles could never pass the tortoise: Suppose that Achilles starts at position a_1 and tortoise starts at position t_1. (See Figure 6.) When Achilles reaches the point $a_2 = t_1$, the tortoise is farther ahead at position t_2. When Achilles reaches $a_3 = t_2$, the tortoise is at t_3. This process continues indefinitely and so it appears that the tortoise will always be ahead! But this defies common sense.

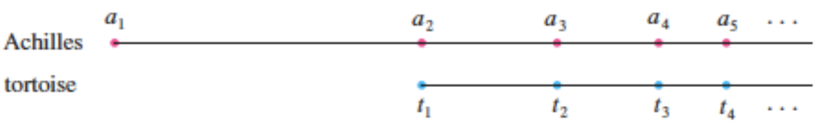


Figure 6

One way of explaining this paradox is with the idea of a *sequence*. The successive positions of Achilles $(a_1, a_2, a_3, \ldots)$ or the successive positions of the tortoise $(t_1, t_2, t_3, \ldots)$ form what is known as a sequence.

In general, a sequence $\{a_n\}$ is a set of numbers written in a definite order.

Example 1. The concept of the limit of a sequence occurs whenever we use the decimal representation of a real number. For instance, if

$$a_1 = 3.1$$
$$a_2 = 3.14$$
$$a_3 = 3.141$$
$$a_4 = 3.1415$$
$$a_5 = 3.14159$$
$$a_6 = 3.141592$$
$$a_7 = 3.1415926$$
$$\vdots$$

then $$\lim_{n\to\infty} a_n = \pi$$

The terms in this sequence are rational approximations to π.

Let's return to Zeno's paradox. The successive positions of Achilles and the tortoise form sequences $\{a_n\}$ and $\{t_n\}$, where $a_n < t_n$ for all n . It can be shown that both sequences have the same limit:

$$\lim_{n\to\infty} a_n = p = \lim_{n\to\infty} t_n$$

It is precisely at this point p that Achilles overtakes the tortoise.

Example 2. Find the limit of $\left\{\frac{sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right\}$.

Solution: If $f(x)$ is a continuous function with $\lim_{x\to\infty} f(x) = L$, and if $a_n = f(n)$ for all values of n then $\{a_n\}$ converges and has the limit value L.
Consider $f(x) = \frac{\sin(x)}{x}$. We know from that as x approaches zero, the function approaches the limit value of one. Hence, since the limit of function $f(x) = \frac{\sin(x)}{x}$ is one, then the sequence converges and has the limit value of one.

Example 3. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution: If we write out the terms of the sequence, we obtain

$$\{-1,1,-1,1,-1,1,\dots\}$$

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n\to\infty}(-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent.

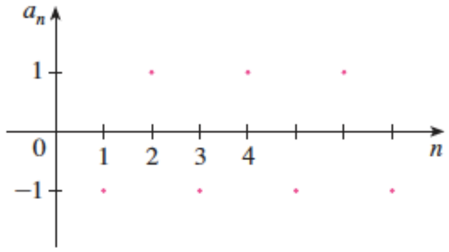


Figure 7

Example 4. Find $\lim_{n\to\infty} sin(\pi/n)$.

Solution: Because the sine function is continuous at 0, Theorem * enables us to write

$$\lim_{n\to\infty} sin(\pi/n) = sin \lim_{n\to\infty} (\pi/n) = sin0 = 0.$$

Remark: Theorem*: If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

III. Derivation
Derivatives rules

Example 1. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Solution: Horizontal tangents occur where the derivative is zero. We have

$$y' = (x^4 - 6x^2 + 4)' = 4x^3 - 12x = 4x(x^2 - 3) = 0$$

Thus $y' = 0$ if $x = 0$ or $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0, \sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0,4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 8.)

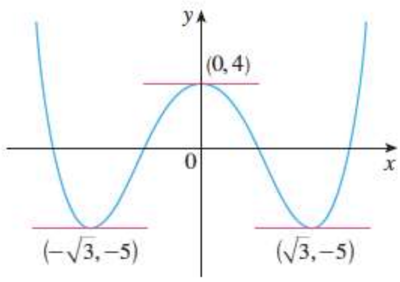


Figure 8

Example 2. The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Solution: The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3, \qquad a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 14\, cm/s^2$.

Example 3. The cost to produce q items is $C(q) = q^2(\sqrt{q} - 2)$ dollars. Find the marginal cost of producing the 25^{th} item. Interpret the answer in terms of costs.

Solution: The marginal cost is defined as $C'(q)$. In this case,

$$C(q) = q^2(\sqrt{q} - 2) = q^2\left(q^{\frac{1}{2}} - 2\right)$$

Then, the marginal cost function is

$$C'(q) = 2q\left(q^{\frac{1}{2}} - 2\right) + q^2 \cdot \frac{1}{2} q^{-\frac{1}{2}} = 2q^{\frac{3}{2}} - 4q + \frac{1}{2} q^{\frac{3}{2}} = \frac{5}{2} q^{\frac{3}{2}} - 4q$$

Hence, the marginal cost of producing the 25^{th} item is

$$C'(25) = \frac{5}{2}(25)^{\frac{3}{2}} - 4 \cdot 25 = 212.5 \text{ dollars/item}$$

The statement $C'(25) = 212.5$ dollars per item says that when 25 items have already made, the cost of making the next item is approximately \$212.5. Another way of saying this is that it costs about \$212.5 to make the 26^{th} item.

Example 4. The sales (in millions of dollars) of a DVD player t years after it is put on the market is given by

$$S(t) = \frac{100t}{1 + e^t}$$

How fast is the sales changing at the end of the first year and how fast at the end of the second year?

Solution: The problem is asking for the rate at which sales are changing at the end of the first year and at the end of second year. The rate at which sales are changing at time t is given by $S'(t)$. Using the quotient rule, we have

$$S'(t) = \frac{d}{dt}\left[\frac{100t}{1 + e^t}\right] = \frac{100(1 + e^t) - 100t \cdot e^t}{(1 + e^t)^2} = \frac{100 + 100e^t(1 - t)}{(1 + e^t)^2}$$

Therefore

$$S'(1) = \frac{100 + 0}{(1 + e^1)^2} = 7.33$$

$$S'(2) = \frac{100+100e^2(1-2)}{(1+e^2)^2} = -9.078$$

Thus, sales are increasing at a rate of \$7.233 million per year at the end of the first year, and sales are decreasing at a rate of \$9.078 million per year at the end of the second year.

IV. The integral
Indefinite integral (anti derivative)

Example 1. Given $g(x) = 4x^3 + 3x^2 - 4x + 1$, show that its antiderivative is $f(x) = x^4 + x^3 - 2x^2 + x + 2$.

Solution: To show that $f(x)$ is the antiderivative of $g(x)$ we need to take the derivative of $f(x) = x^4 + x^3 - 2x^2 + x + 2$.

$$\frac{d}{dx}f(x) = \frac{d}{dx}(x^4 + x^3 - 2x^2 + x + 2) = 4x^3 + 3x^2 - 4x + 1 = g(x),$$

Since $\frac{d}{dx}f(x) = g(x)$, $f(x)$is the antiderivative of $g(x)$.

Recall that the derivative of a function tells us about the slope of a line tangent to the graph of the function at some value $x = a$. The graphs of a few functions of the form $f(x) = x^4 + x^3 - 2x^2 + x + C$ are shown in Figure 1 below.

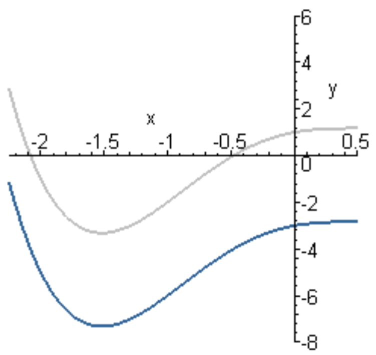


Figure 9: Graphs of $f_1(x) = x^4 + x^3 - 2x^2 + x + 1$ and $f_2(x) = x^4 + x^3 - 2x^2 + x - 3$

Since the shape of each curve is the same and the curves are just translations of one another, we would predict that the slope of the tangent lines at all values of $x = a$ must be the same for $f_1(x)$ and $f_2(x)$. Figure 10 below has the tangent line drawn for both curves at $x = -1$.

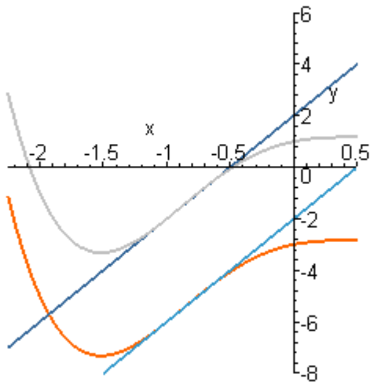


Figure 10

Since the two tangent lines at $x = -1$ are parallel, $f_1(-1) = f_2(-1)$. Thus, it seems safe to conclude that $f_1'(a) = f_2'(a)$ for all values of $x = a$. We can also show this algebraically and see that $f_1'(x) = 4x^3 + 3x^2 - 4x + 1$ and $f_2'(x) = 4x^3 + 3x^2 - 4x + 1$. So, which function, $f_1(x)$ or $f_2(x)$ is the antiderivative of $4x^3 + 3x^2 - 4x + 1$? The answer is that both $f_1(x)$ and $f_2(x)$ are antiderivatives of $4x^3 + 3x^2 - 4x + 1$. As a matter of fact, any function of the form $f(x) = x^4 + x^3 - 2x^2 + x + C$, where C is some constant, is an antiderivative of $4x^3 + 3x^2 - 4x + 1$. Thus, the most **general antiderivative** of $h(x) = 4x^3 + 3x^2 - 4x + 1$ is $H(x) = x^4 + x^3 - 2x^2 + x + C$.

This process of antidifferentiation is also known as **integration**. Integration uses the symbol $\int$ (called the integral sign) to denote that the antiderivative of a function is to be taken. When the integral sign is used we always find the most general antiderivative of the function that immediately follows the integral sign. The arbitrary constant C that comes from finding the most general antiderivative is called the **constant of integration**.

Example 2. Find $f(x)$ given $f'(x) = 4x + 3 + e^x$ if $f(0) = -9$.

Solution: First find

$$\int (4x + 3)\,dx = \frac{4}{2}x^2 + 3x + e^x + C$$

Since $f(0) = -9$, we get

$-9 = 2 \cdot 0 + 3 \cdot 0 + e^0 + C$

$-9 = 1 + C$

$-10 = C$

so $f(x) = 2x^2 + 3x + e^x - 10$.

Example 3. The rate of change in sales of bicycles at Ted's bicycle shop for the year 2001 is given by $s(x) = -x + 7.5$, where x represents the month number in 2001 (i.e. $x = 1$ is January, $x = 2$ is February, … $x = 12$ is December). If the shop sold 28 bicycles in the month of February, find the shop's total sales function, $TS(x)$.

Solution: Since we are given the rate of change of sales, we can find the total sales function, $TS(x)$, by finding $\int s(x)\,dx$.

$$TS(x) = \int s(x)\,dx = \int (-x + 7.5)\,dx = -\int x\,dx + 7.5\int dx = -\frac{1}{2}x^2 + 7.5x + C$$

We also know that $TS(2) = 28$ because the shop sold a total of 28 bicycles in the month of February. Thus,

$28 = -\frac{1}{2}x^2 + 7.5x + C$

$28 = -\frac{1}{2}2^2 + 7.5 \cdot 2 + C$

$28 = 13 + C$

$15 = C$

So the sales function is $TS(x) = -\frac{1}{2}x^2 + 7.5x + 15$.