LEARNING MATHEMATICS:
A TEACHER'S MANUAL

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INTRODUCTION

In this document we present teaching techniques that are recommended for teaching mathematics by partner universities within the iTEM (Innovative Teaching Education in Mathematics) project. These techniques include:

- Active Learning and In Pairs Tasks Solving
- Frequent testing
- PBL (Project-Based Learning) & POPBL (Problem-Oriented Project-Based Learning)
- Visualizations

Our goal is to demonstrate to the students that mathematics is relevant to the program they have chosen to study and that they have the ability to learn mathematics and succeed in learning this subject. The proposed teaching techniques provide students with combined learning strategies to help them to better understand mathematics, to better and more precisely formulate mathematical issues, to more efficiently look for ways to solve problems and increase their capability to apply earned skills and knowledge. These strategies together with particular mathematical topics linked to adequate real-life problems are intended to promote students' understanding and their abilities in logical thinking and creativity.

METHODS OF TEACHING

ACTIVE LEARNING AND IN PAIRS TASK SOLVING [1]

The proposed strategy of teaching involves a combined approach of:

a) Demonstrating relevance to the subject matter

b) Active learning
c) Peer-learning and active task solving in pairs
d) Immediate feedback
e) Visualizations
f) Support for students who fall behind

DEMONSTRATING RELEVANCE TO THE SUBJECT MATTER

Demonstrating relevance to the subject matter is a crucial component in fostering motivation among students and preferably it should be presented as an introduction to the subject learned and not just as a by-product after the mathematical topics have been introduced, taught, and practiced.

When a teacher demonstrates the relevance of a mathematical topic to the subject the students have chosen to specialize in, it becomes more appealing to the students, since this is the field they themselves decided to study. The effort students invest in learning mathematics is not futile. The knowledge they gain will not only suffice to pass mathematics exams, but the effort they devote in learning mathematics will also bare fruits and help them succeed further in their studies as well as their future career. If the teacher is consistent in showing the connection between mathematics and the subject the students want to specialize in, the students will understand that mathematical knowledge is not a foreign element in the curriculum. Rather, mathematics is critical to the understanding of the core courses in the students' desired field of knowledge and to their professional success.

Ideas for relevant problems can be found in books that introduce mathematics from an applied perspective, e.g., [2], [3], [4], and [5]. iTEM project builds a library of such examples for the benefit of mathematics teachers world-wide. A link to the library will be added to this document once the first phase of the library will be completed.

ACTIVE LEARNING

Freeman et. al. [6] defined "Active Learning" as follows:

Active learning engages students in the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert. It emphasizes higher-order thinking and often involves group work.

Please note that the activities are not specified, thus a wide range of activities may be applied. These activities change the role of the student in class from passive to active. They should emphasize higher-order thinking, and allow students to reflect on their own learning. Active learning naturally provokes a student to logical thinking and together with task solving in pairs, which we will discuss in the following sub-section, leads smoothly to discovering logical connections between the known facts and the new ones. Using these principles, the students in class are redirected to active developmental work instead of focusing on strategies that mainly employ memorization, identification of similar problems and repetition of solution strategies the students have already seen.

A meta-analysis of 225 studies that compared active learning vs. traditional lecturing in the context of STEM education [6] found that average exam scores improve by around 6% in active learning classes, and students taking courses with traditional lecture are 1.5 times more likely to fail than those in classes with active learning. Interestingly, the actual activity used did not have a significant influence on students' performance.

PEER-LEARNING AND ACTIVE TASK SOLVING IN PAIRS

Peer learning can be defined as "the acquisition of knowledge and skill through active helping and supporting among status equals or matched companions. It involves people from similar social groupings who are not professional teachers helping each other to learn and learning themselves by so doing" [7], or "the use of teaching and learning strategies in which students learn with and from each other without the immediate intervention of a teacher" [8]. The opportunity to discuss the material and/or solve a problem with their peers, engages students in a different, more proactive, learning. This form of teaching, where students interchangeably alternate between the role of the teacher and the role of the student is engaging and effective.

There are many strategies for peer learning implementation [9], depending on the context in which peer learning is performed.

The context discussed in this document, teaching mathematics to first year students in higher education, is quite constrained. Students arrive to higher education institutions from different backgrounds and some with lack of sufficient knowledge in mathematics and/or basic learning skills. On the other hand, classes tend to be big and the teachers are required to cover a lot of material at a fast pace. These constraints do not leave many options to implement peer learning.

In pairs learning as a kind of peer learning is applicable both within the lecture (100-250 students in a class) and within the recitation (20 – 40 students).

Within the lecture the arrangement is simple - just to ask the students to introduce themselves to their neighbor specified and ask them to work together. This setting is intended to switch the mode from passive attendance of the lecture to active participation during the lecture. Once the lecturer introduces a new topic or a real-life problem, there might emerge questions. These questions – transformed to tasks are directed to the pairs, so the peers in each pair are asked to exchange ideas, to share their thoughts with their peer, to make decision, to give suggestion, and also to formulate answers to the given tasks. The teacher has to decide how, when, and how many times during the lecture such an activity should be included. In the end of the lecture the lecturer might ask to review what the students have just learned, how difficult the topics presented seemed to them etc. Adequate technology and software (Kahoot, for instance) may and is a great support, but simple voting tools are a good help as well.

There is a significant distinction between active performance of the students as well as teachers within the lecture and within the recitation. At the lecture the teacher is presenting topics, the students' active involvement is limited and depends on the teacher’s decision to which extent the proactive intermissions are included. But even a few of these activities may initiate students’ curiosity in the further steps the teacher is supposed or sometimes suggested to follow. In addition and very importantly the immediate feedback may increase students’ confidence in their own abilities.

At the recitations when managing in pairs tasks solving [10] the teacher has to accept several roles: primarily as an organizer, most of the time as an observer, sometimes as an advisor or a moderator, and rarely as a teacher. We recommend to interfere only when major

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misunderstandings or difficulties emerge, not at all to repeat or substitute the role of a lecturer. The students should and are recommended to use relevant study material. They themselves should communicate with their peer and try to find, decide, and formulate together the best to their knowledge solution to the given task/s. This is the arena for exchange of experiences and approaches between two individuals in a pair, but in dependence on the situation, interesting approach should be shared. Wide range of tasks of various nature (theory, practice, application), layout (in dependence on the topic/s) should be provided as much as possible. This approach has another major advantage: when the problem presented is a task the student already knows how to solve, going through the solution process enhances the student's confidence. When a difficult problem is to be solved in a pair and both students try but do not succeed solving it, the very fact that they were trying to solve it increases their curiosity to learn how to overcome the problem. To the contrary to that, once setting a problem and immediately demonstrate how the problem can be solved does not raise the difficulty of actually finding the solution and as a result, the student lacks the motivation to find the solution.

It should be noted that the optimal size of the group of learners is two only for advanced students, whereas groups of three or more have been shown to solve problems beyond the capability of any single individual in the group [11]. However, practically, the size of the group would be determined by class layout constraints and by the ability of students to communicate with one another in the class. Taking these limitations into consideration, splitting students into pairs is more practical.

In pairs task solving has been proven as an ideal complement to active learning facilitating logical thinking [12]. Additionally, the in pairs task solving method provides students who did not understand the material with the opportunity to learn the material from a peer, at least at a level sufficient enough to continue following the lecture.

Communication in horizontal direction: student-to-student, is less stressful for the partners involved than teacher-to-student. And once the students accept this way of learning, they become more confident and independent in solving problems – not only in searching the resources just to find answers to the given questions. This process also builds students' confidence in their ability to solve complex problems, study independently and succeed in mathematics studies. Very important effect of this setting is that it minimizes the students' stress level. Positive atmosphere in the class is the best stimulus for logic thinking as well as for efficiency of learning.

Regardless of the extent to which pair learning is used, good teaching should take into account students' different learning styles, as student's attitudes and motivational factors are individual [13].

In pairs learning is considered as student oriented [19] and is recommended as one of the pro-active learning options. Its impact will be evaluated within iTEM project.

IMMEDIATE FEEDBACK

Very often educators observe that the students have an impression that they have a good grasp of the topic however their impression is based on their learning experience from secondary or high school and is divorced from reality. They are unaware of the different studies in secondary schools and in higher education institutions. Once these students realize their true level of understanding, it is too late to fill in the gap between the required level and their actual knowledge [10].

To confront this challenge, we highly recommend to introduce *frequent testing* so that both students and educators would have an immediate feedback to the true level of understanding of the individual student and the class as a whole.

The hope is that if students realize early in the semester that they are falling behind, they have an opportunity to recover and adapt themselves to the required level of studies.

Writing, performing and grading many tests can be a daunting task for educators. Many institutions lack the manpower and facilities to perform and grade these tests, leaving the lecturer with responsibility for preparing, executing and grading the tests. Moreover, in many cases the only time the teacher may perform these tests is during the lecture, reducing the already limited time for teacher-to-student interaction.

We, therefore, suggest to build these tests in a learning management system, such as Moodle, and ask the students to perform the tests at least eight times during a three-month semester. These tests should be automated, short and include simple technical questions. Remember that the aim of the tests is not to provide a comprehensive feedback for the student's understanding of the material, but rather to find out whether the student understands basic concepts and can follow the class and recitation.

Students do not tend to perform tests voluntarily. Therefore, we suggest to give those frequent tests weight in determining final grade. The weight in the final score should be small, we suggest that the weight would be somewhere between 10% and 15% of the final grade.

Automated tests may include multiple-choice questions to be graded automatically by platforms such as Moodle. However, whenever possible, we recommend using tools such as STACK [14]. These tools allow the student to provide a mathematical answer rather than choosing the most appropriate answer from a list of pre-defined answers. The tools can then check whether the answer the student provides is algebraically equivalent to the correct answer.

iTEM project builds a library of multiple-choice and STACK-based questions under the Moodle platform. A link will be provided once the first phase of this online library is available.

VISUALIZATIONS

Arcavi [15] provided one of the many definition for visualization:

> "*Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding*"

Students differ in their ability to understand, visualize, and interpret the material learned. However, math tests and curriculum favor the non-visualizers [16]. Visualizations may help students who learn through visual imagery to understand and grasp math. It is recommended to use them as much as possible. The advent of computer graphics and computer animation enable us to create rich representations that can be manipulated by students so that they better understand the material learned. Indeed, Palais [17] stated that "applied mathematicians find that the highly interactive nature of the images produced by recent mathematical visualization software allows them to do mathematical experiments with an ease never before possible". However, visualizations serve as prototypes for mental imagery, and as such they create visual–perceptual limitations, that in turn can affect students' understanding [18].

iTEM project builds an online visualizations library. A link will be provided once the first phase of this online library is available.

SUPPORT FOR STUDENTS WHO FALL BEHIND

The response to the challenge, which students who fall behind pose to the higher education system, can be divided into two main parts: (1) early identification of students who fall behind, and (2) adequate help to students who fall behind.

IDENTIFYING STUDENTS WHO FALL BEHIND

The ability to reflect to students that they fall behind as soon as possible is a key for having some leverage that would allow students in choosing their response to the hurdles they face. The problem is that in the higher education system there are few mechanisms that provide students with a feedback about their performance and the feedback mechanisms that exist (e.g., homework) do not succeed in bringing home the message. As a result, students are often unaware to the fact the they are falling behind, and for many of those students the first time they realize that they are in trouble is close to the final exam, which most of the time is too late to recover the deficit they have already accumulated. Even if they manage to pass the final and the course, this deficit will persist, and will affect their performance later in their studies. Thus, the ability to identify students who fall behind at the early stages of the semester may help these students to recover and close the gap between their level and the expected level of the class.

There are many tools that can be applied in order to provide feedback to the students. Frequent testing, which we suggested earlier, is one of these tools. Students who consistently do not succeed in these tests will have a feedback and may realize they are falling behind.

Additionally, the teaching staff can use the information collected and stored in learning management systems (e.g., Moodle). Processing and translating this information into behavioral patterns may provide a clue as to the level of the individual student. An example of a behavioral pattern that may be interpreted as a behavior of a student at risk is of a student that reduces the frequency of accessing the course environment in the learning management system as the semester progresses. Obviously, there could be other reasons for this behavior, for example, the student participates in a study group and another student in the group is accessing the course material on behalf of the whole group. However, this information coupled with the student's behavior and performance at the frequent testing (e.g., the final score, the number of trials until the student answers a question, the time it took the student to answer each question compared with the rest of the class) may provide a clue that identifies students who are at risk of falling behind.

In the context of the iTEM project we are developing an automatic tool that analyzes information from the Moodle learning management system and provides a list of the students at risk of falling behind. A link to the tool will be provided in future versions of this manual once the development of the first version of the software is available.

Once students are flagged as being at risk of falling behind, there is a need to contact these students and check whether the indications reflect reality. Different schools can apply different methods to access these students depending on their staff and financial abilities. The easiest and less effective way is to alert students electronically via an e-mail. However, a more effective way would be to contact students personally and understand whether they are falling behind and if so, what are the reasons for it. Reasons for falling behind are varied and might be related to external constraints. Therefore, if possible, it is recommended to designate a staff member that is not a part of the teaching team of the student (can be part of the administrative staff) to contact students at risk of falling behind.

In the next sub-section we provide some common reasons for falling behind and suggest potential mitigation options. Obviously, we cannot provide an exhaustive list, as the hurdles students face may vary from one country to another and from one student to another. Discretion is advised as to the individual response to each student.

The reasons students fall behind may vary and are different from one institution to another. However, the main reasons are:

1. Deficit that originates in the secondary school years, that is a lack of prior knowledge
2. Improper study habits, e.g., studying in a similar manner as in high school, failing to grasp the effect of the different expectations of a secondary school compared with a higher education institution
3. A need for slowly paced exercises
4. Lack of time due to:
    a. An overload that stems from taking too many classes
    b. Financial difficulties that force students to spend a significant time working to cover their costs
    c. Problems at home that reduce the ability of the student to study
5. Disabilities

Each one of these reasons requires a different approach. However, we provide some suggestions how to confront them. Staff from each institution will need to come up with a response suitable to local conditions and available resources:

1. Some of the students come to the higher education institution with a lack of required knowledge of mathematics. The reasons vary from one country to another: inequality in the level of schools in different regions, a break that students took between high school and academic life, or any other reason. The end result is that some, if not many, of the students lack prior knowledge. Preferably, students can be directed to take a refresh workshop that will help them close the gap. We also recommend to perform a baseline pre-testing of all students in order to detect students who lack prior knowledge as early as possible in the semester. If students did not participate in this workshop or the workshop was not sufficient, the teaching staff should refer the students to a material available at the web that will help them to study this material on their own. Ideally, the material should be concise and to the level needed to understand the material discussed in class. This material can be texts, videos, etc. Referring students to specific material that will help them learn the prior material they

lack should be done explicitly. For example, specify the prior knowledge that is required in the lecture handout and provide link(s) to material that will help students to close the gap they have in their knowledge. The iTEM project collects material that may help students who lack prior knowledge and a link to this material will be provided in future versions of this manual.

2. Students who lack proper study habits can be referred to a workshop dedicated to combat this problem. While a study habit workshop provides students with tools to better manage themselves, they seldom tackle challenges related to studying a specific subject, such as mathematics. As part of the iTEM project, we have developed a study manual to help students to study mathematics. The study manual is available at the iTEM website.
3. Students should be referred to slowly paced exercises that will help them master the material. The ITEM project will provide such exercises for the first course in Calculus and the first course in Linear Algebra.
4. Students who lack time from various reasons should be consulted to see whether it is possible to allow them to have more time (e.g., refer students who don't have sufficient financial resources to a scholarship). If it is not possible to overcome the time constraints in a very short period of time, students should be advised to reduce their class load for the semester.
5. Students with disabilities should be referred to local facilities available at each institution.

A SUMMARY OF ITEM'S PROPOSED INGREDIENTS FOR MATH CLASSES

The teaching & learning process involves lectures, recitations, and frequent testing.

- Lectures may be conducted in a standard manner. Presentations of individual topics accompanied with adequate examples. The examples can be purely mathematical ones and relevant real-life problems. Preferably the real-life problem would be from the content domain of the main subject the students learn and at least one of the examples should be provided as a prelude to the topic learnt. Visualization is supposed to be an integral part of presentations. As mentioned above to activate students' attention during the lectures in pairs setting may be used and convenient quizzes and/or control questions should be used as well using tools, such as Kahoot.

- Recitations should be structured and should comprise of:

I. Short introduction to the topics (teacher)

II. T - Theoretical tasks (active in pairs learning)

P - Practical tasks (active in-pairs tasks solving)

R - Review of the tasks (Students with the teacher)

Parts {T - P - R} may repeat in dependence of the content of individual recitation

III. Evaluation (Students)

- Frequent testing

  Provided by the institution and it has to be an integral part of students' duties. Adequate support to students who lack prior knowledge and those detected as potentially at risk.

ADVANTAGES AND DISADVANTAGES OF THE ITEM'S APPROACH

The common advantages of the approach comprise of:

- Students learn to formulate, express and write down mathematical statements in accordance with the correct mathematical terminology.
- Reducing dependence of students on their teacher when solving mathematical tasks.
- Changing the pace of the recitation.
- Prevents a passive presence of the students at recitations.

Disadvantages of the approach are:

- The teaching requires a different preparation of the content and slightly different form of lecturing and requires various roles of a teacher during the recitations.
- Some students might be hard to convince to cooperate with others and individual approach should be applied to involve such students.

Conditions to make the method work well:

- The students should have adequate baseline knowledge of mathematics. We recommend baseline pre-testing of all students in order to detect students who lack prior knowledge as early as possible in the semester.
- Students are supposed to study mathematics independently as a preparation for the recitations.

A note of caution: The use of information technology tools (ICT) in education is often viewed as a deus-ex-machina that will also save mathematics education. However, our experience shows that contrary to other disciplines, where ICT facilitate the learning processes undoubtedly very well, their effect in teaching mathematics may be both positive and negative depending on choice of the right teaching strategy and its extent [10].

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