MOTIVATION FOR FORMAL DEFINITION OF LIMITS

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From the informal definition of the limit we know that:

$$\lim_{x \to \pi} x^2 = \pi^2$$

This is because the function x^2 is an elementary function, and it is continuous for all values of x. We can, therefore, conclude that for any x, the limit equals to the function value at x.

It seems simple and straightforward, but what is the value of π^2? We cannot provide the exact value of π, because π is an irrational number. As a result, we cannot compute the exact value of π^2. We will now show an example of a problem in which the precision of the calculation of π^2 is crucial. We will present the problem of calculating the orbit of a planet around the sun. Specifically, we will attempt to calculate the orbit of planet Mercury around the sun.

The average radius of Mercury's orbit around the sun is: 5.8×10^{10} meters. The mass of the sun is: 1.99×10^{30} kilograms. We can use Kepler's third law to determine the time it takes planet Mercury to complete a single orbit around the sun. This law states that the orbital time of a planet, T_M, holds the following equation:

$${T_M}^2 = \left(\frac{4\pi^2}{Gm_s}\right) r^3$$

Where m_s denotes the mass of the sun (which is known to us), r is the average radius of the orbit of planet Mercury around the sun (also known to us) and G is a constant of the solar system whose value is:

$$G = 6.673 \times 10^{-11} \left[\frac{N \cdot m^2}{kg}\right]$$

As one can observe, the accuracy of the calculation planet Mercury's orbit[1] depends on the accuracy of the value we use for π^2.

Suppose we require to calculate the value of π^2 to an accuracy of 0.01. This requirement is translated to the following question: what is the accuracy of π needs to be to give us an accuracy of 0.01 in the calculation of π^2. We can rephrase this question slightly: how close do we need to get to the true value of π in order to get close enough to the true value of π^2 to meet the accuracy requirements of the problem.

The formal definition of the limit provides answer to this question.

We want the distance between the calculated value, x^2, and the value of π^2 to be less than 0.01. How far from the exact value of π be from the actual value of π to meet this requirement? We have the tools to formulate this problem mathematically.

How do we measure distance from a point? With the notion of an absolute value. So, we can phrase our problem like this: If we want $|x^2 - \pi^2| < 0.01$, how small should the difference be $|x - \pi|$?

Further in our studies, we will learn how to solve such a problem, but its solution is not trivial. At this point, we will re-formulate the problem more simply by taking root from both sides of the equation, which describes Kepler's third law. If we do this we will get:

$$T_M = 2\pi \cdot \sqrt{\frac{r^3}{Gm_s}}$$

For this calculation we need to find an approximation for 2π. We can formulate our accuracy problem as follows: If we require that $|2x - 2\pi| < 0.01$, how small $|x - \pi|$ should be? We can find the solution as follows:

$$|2x - 2\pi| = 2 \cdot |x - \pi| < 0.01$$

Our requirement is that $|2x - 2\pi| < 0.01$, and we succeeded to express it as a function of what we are looking for: $|x - \pi|$. So we can say that:

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[1] Please note that the other values are also approximations and not exact values. What we are trying to do in this exercise it to match the level of accuracy of the value of π^2 with the rest of the values in the formula.

$$|x-\pi| < \frac{0.01}{2} = 0.005$$

In other words, if the distance of x from π is less than 0.005, then the distance of $2x$ from 2π is less than 0.01, in accordance with our accuracy requirements.

Another way to do find the when we comply with the accuracy requirements is using absolute values, as follows:

$$2\pi - 0.01 < 2x < 2\pi + 0.01 \iff |2x - 2\pi| < 0.01$$

And from this expression it follows that:

$$\pi - 0.005 < x < \pi + 0.005$$

We will now demonstrate this process graphically. First, we will observe how the graph of the function $y = 2x$ looks like and the area around π in the x-axis and around 2π in the y-axis.

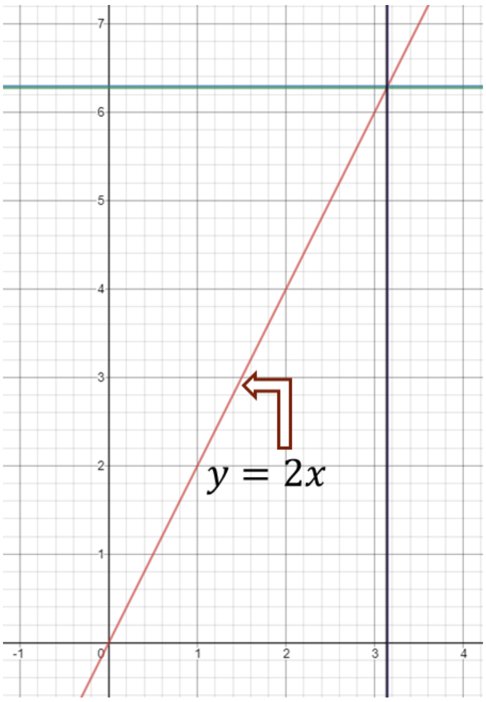

Figure1 : the graph of the function $y = 2x$

Now let's have a close-up look at the graph around π on the x axis and around 2π on the y axis.

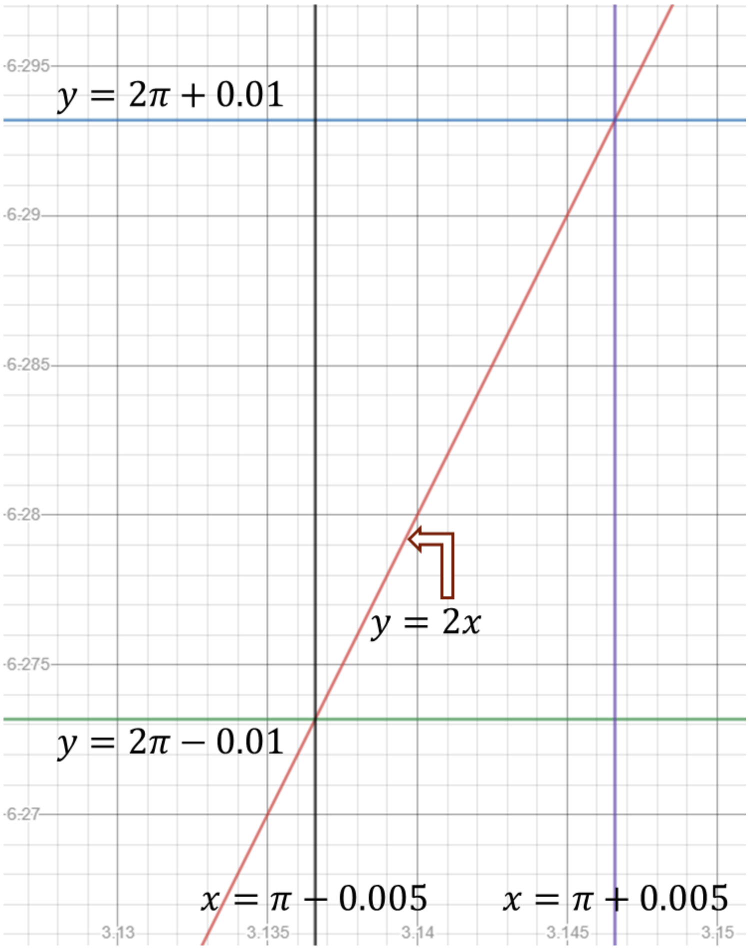

Figure 2: a close-up look at the graph around π on the x axis and around 2π on the y axis

Between the blue and green lines we find the permitted range for the value of the 2π calculation so that it remains within the accuracy requirements we have defined, between $2\pi + 0.01$ and $2\pi - 0.01$. To reach this range, we must use the value of π that is between the black line and purple line, or between $\pi + 0.005$ and $\pi - 0.005$. We can describe it mathematically as follows:

$$|2x - 2\pi| < 0.01 \iff |x - \pi| < 0.005$$

The formal definition of the limit solves a similar problem. We can define ε as the precision requirement of the calculation and set δ to be the answer that guarantees the accuracy requirement. The formal definition of the limit:

$$\lim_{x \to a} f(x) = L$$

Means that for every $\varepsilon > 0$ there is $\delta > 0$ so that every x in the function range exists:

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

We can rephrase this statement to conform with the accuracy problem: for the requested accuracy, $\varepsilon > 0$, we are looking for $\delta > 0$ to ensures that if: $x \in (a - \delta, a + \delta) - \{0\}$ then the function $f(x)$ will be within the requested precision range: $f(x) \in (L - \varepsilon, L + \varepsilon)$.

In Figure 3 you can observe an example of a formal calculation of the limit:

$$\lim_{x \to 2} x^2 = 4$$

Given an accuracy requirement, some accuracy, ε, greater than 0. We look for $\delta > 0$ so that for every x in the definition field exists:

$$0 < |x - 2| < \delta \implies |x^2 - 4| < \varepsilon$$

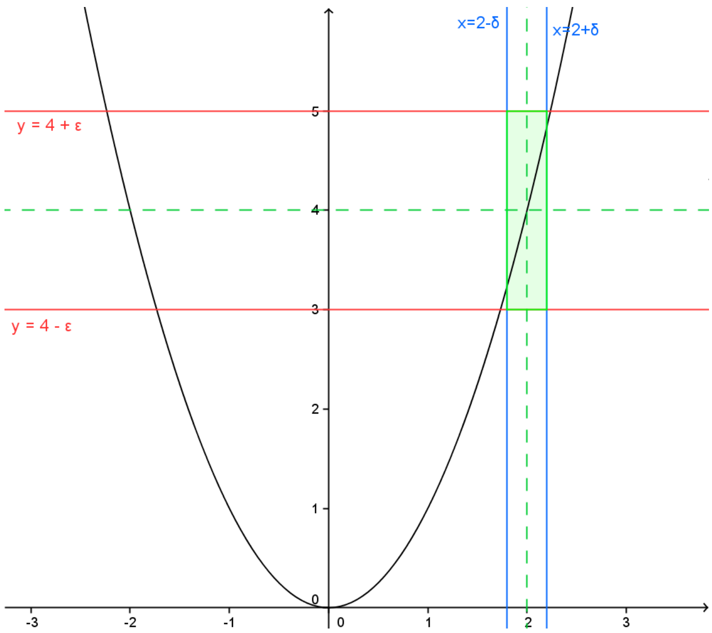

Figure 3: example of a formal calculation of the limit: $\lim_{x \to 2} x^2 = 4$