Reporting Examples from Technology and Real Life Problems within the Curricula of Linear Algebra and Calculus I

Course: Calculus I

Topic: Sketch of the function/ Function analysis

Abstract: Many real life problems become to draw a graph of a function. We provide below some representative examples.

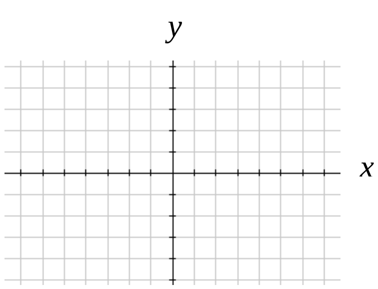
Examples: Optimization problems of one variable

Prerequisites: Graph of the one variable function

Graphs as Functions

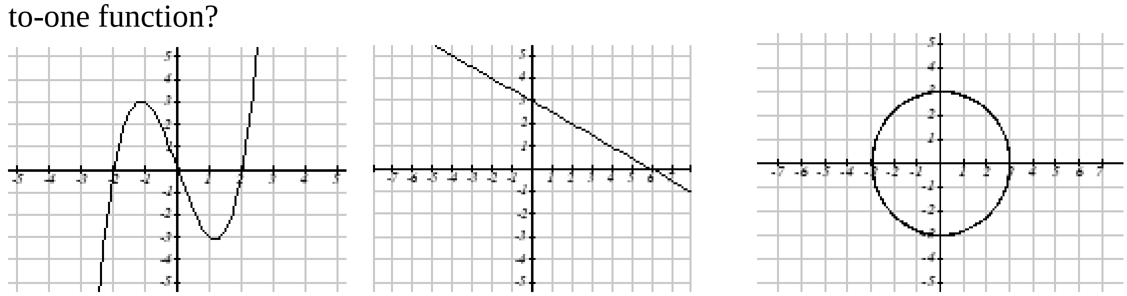
Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

The most common graph has *y* on the vertical axis and *x* on the horizontal axis, and we say *y* is a function of *x*, or *y* = *f(x)* when the function is named *f*.



Example 1

Which of these graphs defines a function *y=f(x)*? Which of these graphs defines a one-to-one function?



Looking at the three graphs above, the first two define a function $y=f(x)$, since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the y-value of the graph. The 3rd graph does not define a function $y=f(x)$ since some input values, such as $x=2$, correspond with more than one output value.

Graph 1 is not a one-to-one function. For example, the output value 3 has two corresponding input values, -2 and 2.3

Graph 2 is a one-to-one function; each input corresponds to exactly one output, and every output corresponds to exactly one input.

Graph 3 is not even a function so there is no reason to even check to see if it is a one-to-one function.

Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

Horizontal Line Test

Once you have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line crosses the graph more than once, then the graph does not define a one-to-one function.

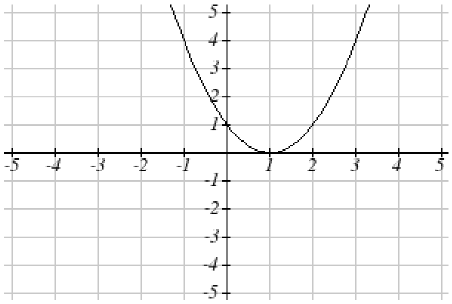
Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 2

Given the graph below,

a) Evaluate $f(2)$

b) Solve $f(x) = 4$



a) To evaluate $f(2)$, we find the input of $x=2$ on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of $y=1$. So $f(2) = 1$

b) To solve $f(x) = 4$, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1,4) and (3,4). These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$
This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4.

Notice that while the graph in the previous example is a function, getting two input values for the output value of 4 shows us that this function is not one-to-one.

Try it Now

5. Using the graph from example 9, solve $f(x)=1$.

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 3

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$ if possible.

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as p = [something involving n].

$2n + 6p = 12$	subtract $2n$ from both sides
$6p = 12 - 2n$	divide both sides by 6 and simplify

$$p = \frac{12 - 2n}{6} = \frac{12}{6} - \frac{2n}{6} = 2 - \frac{1}{3}n$$

Having rewritten the formula as $p=$, we can now express p as a function:

$$p = f(n) = 2 - \frac{1}{3}n$$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations - no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

Example 4

Express the relationship $x^2 + y^2 = 1$ as a function $y = f(x)$ if possible.

If we try to solve for *y* in this equation:

$y^2 = 1 - x^2$

$y = \pm\sqrt{1 - x^2}$

We end up with two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

Example 5

Given the function $k(t) = t^3 + 2$

a) Evaluate $k(2)$

b) Solve $k(t) = 1$

a) To evaluate $k(2)$, we plug in the input value 2 into the formula wherever we see the input variable *t*, then simplify

$k(2) = 2^3 + 2$

$k(2) = 8 + 2$

So $k(2) = 10$

b) To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output

$k(t) = 1$ substitute the original formula $k(t) = t^3 + 2$

$t^3 + 2 = 1$ subtract 2 from each side

$t^3 = -1$ take the cube root of each side
$t = -1$

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if $k(t) = 1$ is true when $t = -1$.

$k(-1) = (-1)^3 + 2$
$= -1 + 2$
$= 1$ which was the desired result.

Example 6

Given the function $h(p) = p^2 + 2p$

a) Evaluate *h*(4)
b) Solve *h(p)* = 3

To evaluate *h*(4) we substitute the value 4 for the input variable *p* in the given function.

a) $h(4) = (4)^2 + 2(4)$
= 16 + 8
= 24

b) *h(p)* = 3 Substitute the original function $h(p) = p^2 + 2p$
$p^2 + 2p = 3$ This is quadratic, so we can rearrange the equation to get it = 0
$p^2 + 2p - 3 = 0$ subtract 3 from each side
$p^2 + 2p - 3 = 0$ this is factorable, so we factor it
$(p+3)(p-1) = 0$

By the zero factor theorem since $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0) and so we solve both equations for *p*, finding *p* = -3 from the first equation and *p* = 1 from the second equation.

This gives us the solution: *h(p)* = 3 when *p* = 1 or *p* = -3

We found two solutions in this case, which tells us this function is not one-to-one.

Try it Now

6. Given the function $g(m) = \sqrt{m-4}$

a. Evaluate *g*(5)
b. Solve *g(m)* = 2

Basic Toolkit Functions

In this text, we will be exploring functions – the shapes of their graphs, their unique features, their equations, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of elements to build from. We call these our "toolkit of functions" – a set of basic named functions for which we know the graph, equation, and special features.

For these definitions we will use x as the input variable and $f(x)$ as the output variable.

Toolkit Functions

Linear

Constant:	$f(x) = c$, where c is a constant (number)
Identity:	$f(x) = x$
Absolute Value:	$f(x) = \lvert x \rvert$

Power

Quadratic:	$f(x) = x^2$
Cubic:	$f(x) = x^3$
Reciprocal:	$f(x) = \frac{1}{x}$
Reciprocal squared:	$f(x) = \frac{1}{x^2}$
Square root:	$f(x) = \sqrt[2]{x} = \sqrt{x}$
Cube root:	$f(x) = \sqrt[3]{x}$

You will see these toolkit functions, combinations of toolkit functions, their graphs and their transformations frequently throughout this book. In order to successfully follow along later in the book, it will be very helpful if you can recognize these toolkit functions and their features quickly by name, equation, graph and basic table values.

Not every important equation can be written as $y = f(x)$. An example of this is the equation of a circle. Recall the distance formula for the distance between two points:

$$dist = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

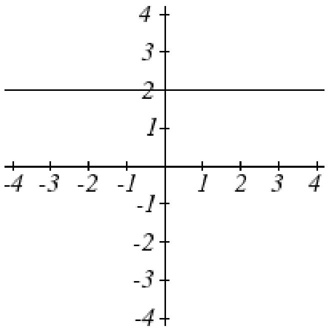
A circle with radius r with center at (h, k) can be described as all points (x, y) a distance of r from the center, so using the distance formula, $r = \sqrt{(x-h)^2 + (y-k)^2}$, giving

Equation of a circle

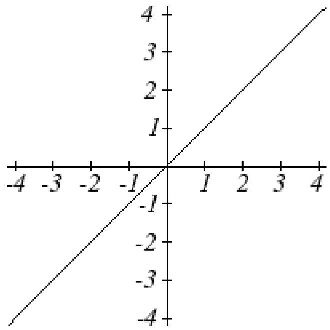
A circle with radius r with center (h, k) has equation $r^2 = (x-h)^2 + (y-k)^2$

Graphs of the Toolkit Functions

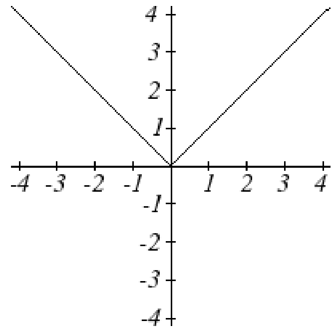
Constant Function: $f(x) = 2$



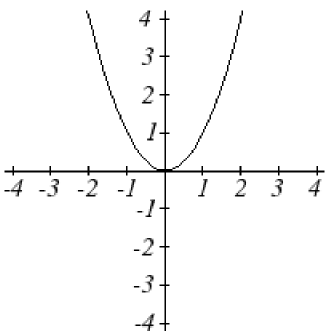
Identity: $f(x) = x$



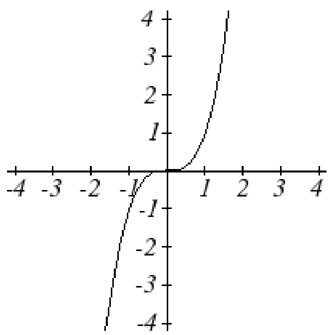
Absolute Value: $f(x) = |x|$



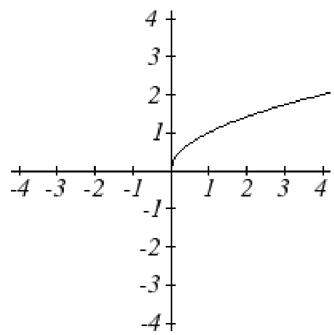
Quadratic: $f(x) = x^2$



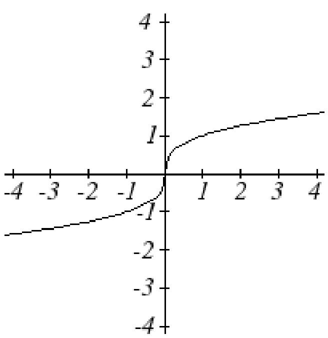
Cubic: $f(x) = x^3$



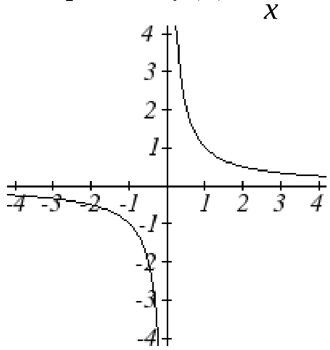
Square root: $f(x) = \sqrt{x}$



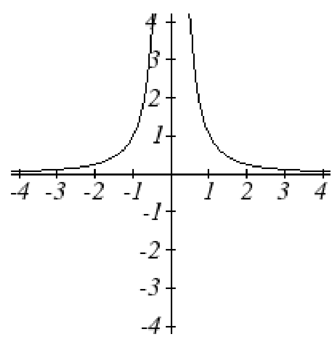
Cube root: $f(x) = \sqrt[3]{x}$



Reciprocal: $f(x) = \frac{1}{x}$



Reciprocal squared: $f(x) = \frac{1}{x^2}$



Important Topics of this Section

Definition of a function
Input (independent variable)
Output (dependent variable)

Definition of a one-to-one function
Function notation
Descriptive variables
Functions in words, tables, graphs & formulas
Vertical line test
Horizontal line test
Evaluating a function at a specific input value
Solving a function given a specific output value
Toolkit Functions

Try it Now Answers

1. Yes
2. No
3. Yes it's a function; No, it's not one-to-one
4. $Q=g(4)=6$
5. $x = 0$ or $x = 2$
6. a. $g(5)=1$ b. $m = 8$

Section 1.1 Exercises

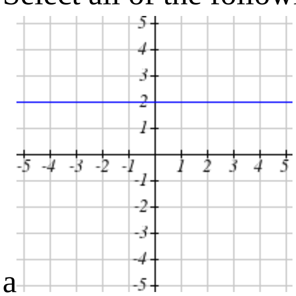
1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.
    a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
    b. Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.
    a. A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.
    b. Explain the meaning of the statement $g(100) = 1$.

3. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
    a. $f(5) = 30$ b. $f(10) = 40$

4. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
    a. $h(1) = 200$ b. $h(2) = 350$

5. Select all of the following graphs which represent y as a function of x.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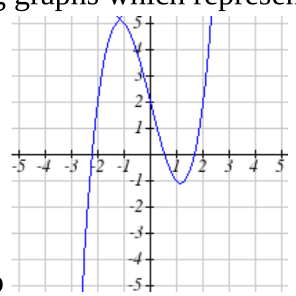
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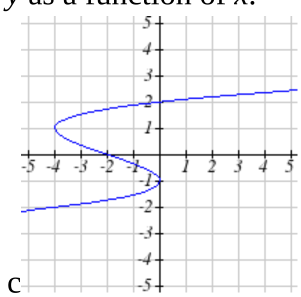
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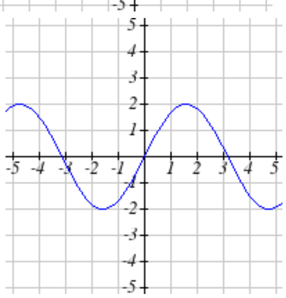
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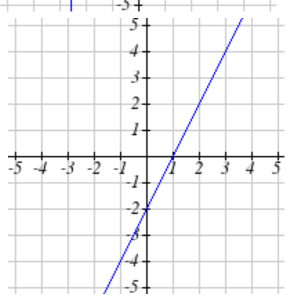
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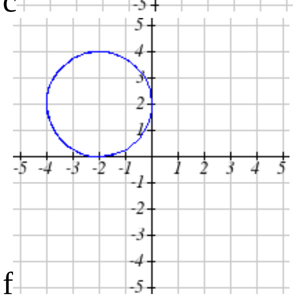
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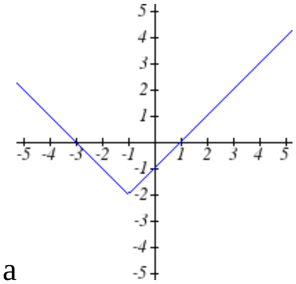
6. Select all of the following graphs which represent *y* as a function of *x*.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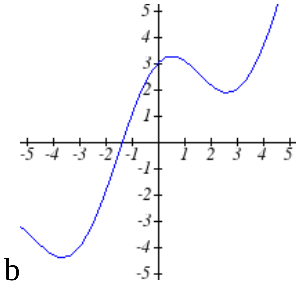
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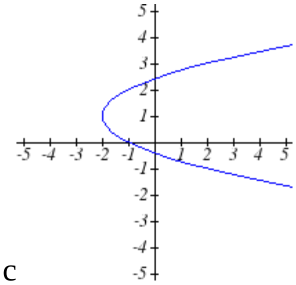
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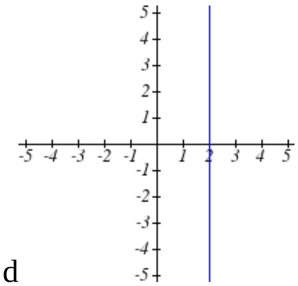
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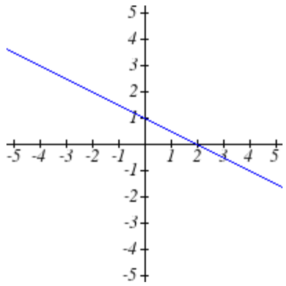
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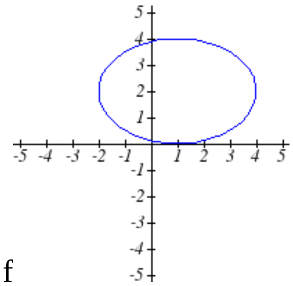
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7. Select all of the following tables which represent *y* as a function of *x*.

a.

x	5	10	15
y	3	8	14

b.

x	5	10	15
y	3	8	8

c.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent *y* as a function of *x*.

a.

x	2	6	13
y	3	10	10

b.

x	2	6	6
y	3	10	14

c.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent *y* as a function of *x*.

a.

x	***y***
0	-2
3	1
4	6
8	9
3	1

b.

x	***y***
-1	-4
2	3
5	4
8	7
12	11

c.

x	***y***
0	-5
3	1
3	4
9	8
16	13

d.

x	***y***
-1	-4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent *y* as a function of *x*.

a.

x	***y***
-4	-2
3	2
6	4
9	7
12	16

b.

x	***y***
-5	-3
2	1
2	4
7	9
11	10

c.

x	***y***
-1	-3
1	2
5	4
9	8
1	2

d.

x	***y***
-1	-5
3	1
5	1
8	7
14	12

11. Select all of the following tables which represent y as a function of x **and** are one-to-one.

a.

x	3	8	12
y	4	7	7

b.

x	3	8	12
y	4	7	13

c.

x	3	8	8
y	4	7	13

12. Select all of the following tables which represent y as a function of x **and** are one-to-one.

a.

x	2	8	8
y	5	6	13

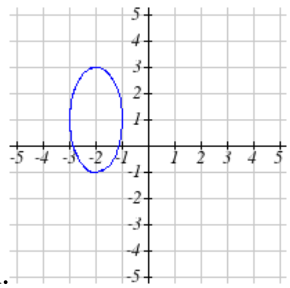
b.

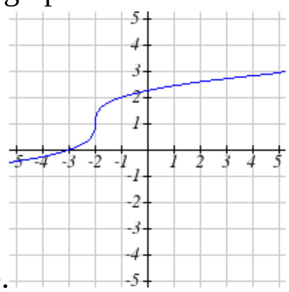
x	2	8	14
y	5	6	6

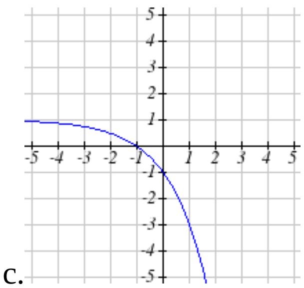
c.

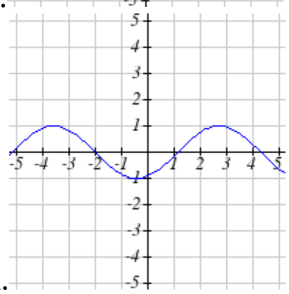
x	2	8	14
y	5	6	13

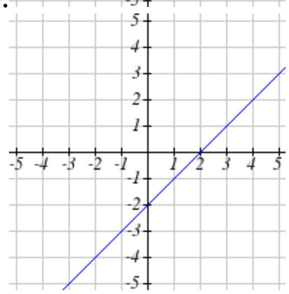
13. Select all of the following graphs which are **one-to-one functions**.

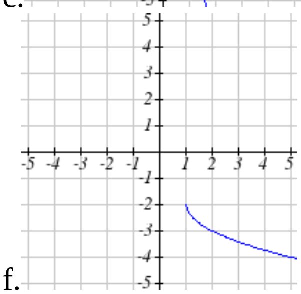
a. 

b. 

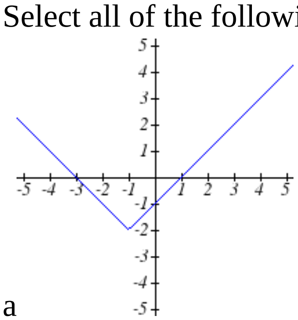
c. 

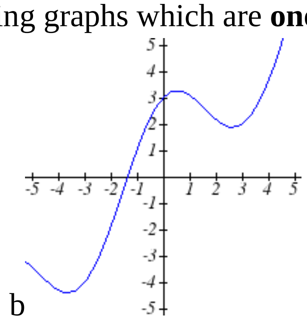
d. 

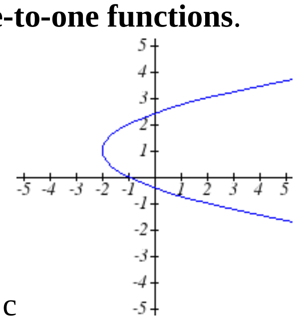
e. 

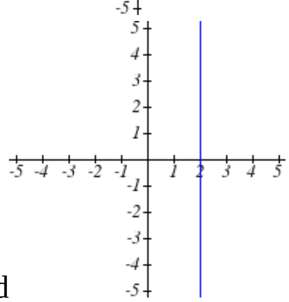
f. 

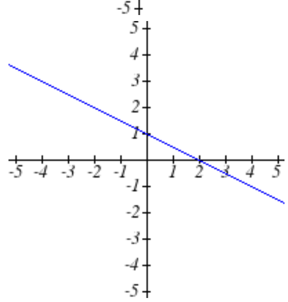
14. Select all of the following graphs which are **one-to-one functions**.

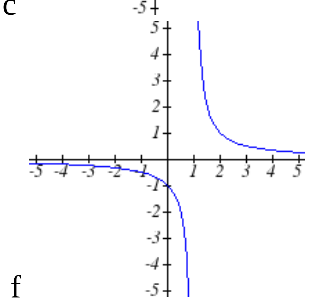
a 

b 

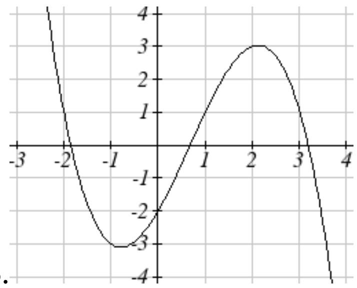
c 

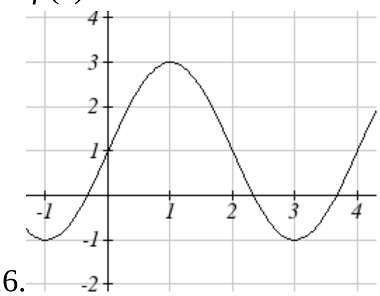
d 

e 

f 

Given each function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$

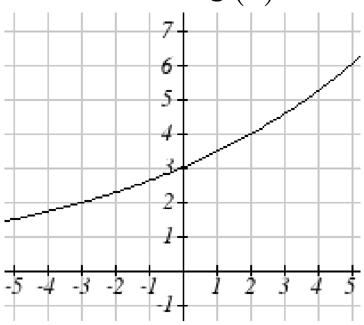
15. 

16. 

17. Given the function $g(x)$ graphed here,

a. Evaluate $g(2)$

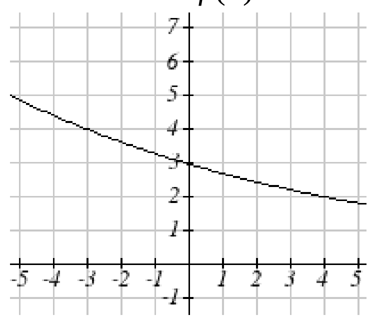
b. Solve $g(x)=2$



18. Given the function $f(x)$ graphed here.

a. Evaluate $f(4)$

b. Solve $f(x)=4$



19. Based on the table below,

a. Evaluate $f(3)$ b. Solve $f(x)=1$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

20. Based on the table below,

a. Evaluate $f(8)$ b. Solve $f(x)=7$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	62	8	7	38	86	73	70	39	75	34

For each of the following functions, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$

21. $f(x)=4-2x$

22. $f(x)=8-3x$

23. $f(x)=8x^2-7x+3$

24. $f(x)=6x^2-7x+4$

25. $f(x)=-x^3+2x$

26. $f(x)=5x^4+x^2$

27. $f(x)=3+\sqrt{x+3}$

28. $f(x)=4-\sqrt[3]{x-2}$

29. $f(x)=(x-2)(x+3)$

30. $f(x)=(x+3)(x-1)^2$

31. $f(x)=\dfrac{x-3}{x+1}$

32. $f(x)=\dfrac{x-2}{x+2}$

33. $f(x)=2^x$

34. $f(x)=3^x$

35. Suppose $f(x) = x^2 + 8x - 4$. Compute the following:
   a. $f(-1) + f(1)$ b. $f(-1) - f(1)$

36. Suppose $f(x) = x^2 + x + 3$. Compute the following:
   a. $f(-2) + f(4)$ b. $f(-2) - f(4)$

37. Let $f(t) = 3t + 5$
   a. Evaluate $f(0)$ b. Solve $f(t) = 0$

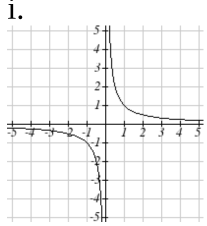
38. Let $g(p) = 6 - 2p$
   a. Evaluate $g(0)$ b. Solve $g(p) = 0$

39. Match each function name with its equation.

a.	$y = x$	i.	Cube root
b.	$y = x^3$	ii.	Reciprocal
c.	$y = \sqrt[3]{x}$	iii.	Linear
d.	$y = \frac{1}{x}$	iv.	Square Root
e.	$y = x^2$	v.	Absolute Value
f.	$y = \sqrt{x}$	vi.	Quadratic
g.	$y = \lvert x \rvert$	vii.	Reciprocal Squared
h.	$y = \frac{1}{x^2}$	viii.	Cubic

40. Match each graph with its equation.

a. $y = x$
b. $y = x^3$
c. $y = \sqrt[3]{x}$
d. $y = \frac{1}{x}$
e. $y = x^2$
f. $y = \sqrt{x}$
g. $y = \lvert x \rvert$
h. $y = \frac{1}{x^2}$

i.

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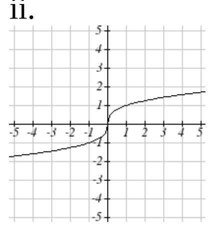
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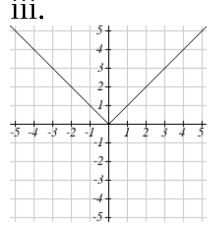
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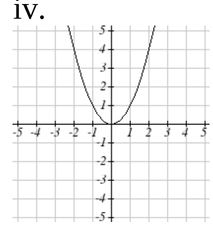
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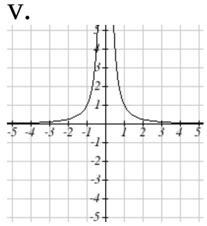
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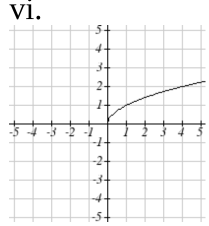
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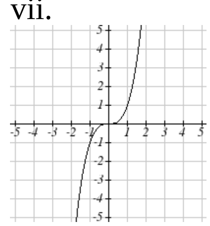
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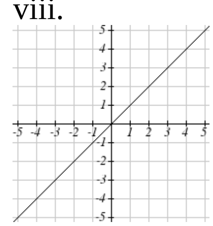


41. Match each table with its equation.

a. $y = x^2$
b. $y = x$
c. $y = \sqrt{x}$
d. $y = 1/x$
e. $y = |x|$
f. $y = x^3$

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

42. Match each equation with its table

a. Quadratic
b. Absolute Value
c. Square Root
d. Linear
e. Cubic
f. Reciprocal

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

43. Write the equation of the circle centered at $(3, -9)$ with radius 6.

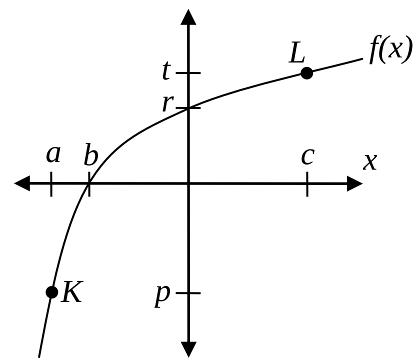
44. Write the equation of the circle centered at $(9, -8)$ with radius 11.

45. Sketch a reasonable graph for each of the following functions. [UW]

a. Height of a person depending on age.
b. Height of the top of your head as you jump on a pogo stick for 5 seconds.
c. The amount of postage you must put on a first class letter, depending on the weight of the letter.

46. Sketch a reasonable graph for each of the following functions. [UW]
    a. Distance of your big toe from the ground as you ride your bike for 10 seconds.
    b. Your height above the water level in a swimming pool after you dive off the high board.
    c. The percentage of dates and names you'll remember for a history test, depending on the time you study.

47. Using the graph shown,
    a. Evaluate $f(c)$
    b. Solve $f(x) = p$
    c. Suppose $f(b) = z$. Find $f(z)$
    d. What are the coordinates of points L and K?



48. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s = d(t)$ which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes." Assume Dave's path to Gould Hall is long a straight line which is 2400 feet long. [UW]

    a. Dave leaves Padelford Hall and walks at a constant spend until he reaches Gould Hall 10 minutes later.

    b. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.

    c. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.

d. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.

e. Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of “distance from Padelford” vs. “time” for the both Angela and Dave.

f. Suppose you want to sketch the graph of a new function s = g(t) that keeps track of Dave’s distance s from Gould Hall at time t. How would your graphs change in (a)-(e)?

Determine the vertical asymptote(s) of each function. If none exists, state that fact.

1. $f(x) = \dfrac{2x - 3}{x - 5}$

2. $f(x) = \dfrac{x + 4}{x - 2}$

3. $f(x) = \dfrac{3x}{x^2 - 9}$

4. $f(x) = \dfrac{5x}{x^2 - 25}$

5. $f(x) = \dfrac{x + 2}{x^3 - 6x^2 + 8x}$

6. $f(x) = \dfrac{x + 3}{x^3 - x}$

7. $f(x) = \dfrac{x + 6}{x^2 + 7x + 6}$

8. $f(x) = \dfrac{x + 2}{x^2 + 6x + 8}$

9. $f(x) = \dfrac{6}{x^2 + 36}$

10. $f(x) = \dfrac{7}{x^2 + 49}$

Determine the horizontal asymptote of each function. If none exists, state that fact.

11. $f(x) = \dfrac{6x}{8x + 3}$

12. $f(x) = \dfrac{3x^2}{6x^2 + x}$

13. $f(x) = \dfrac{4x}{x^2 - 3x}$

14. $f(x) = \dfrac{2x}{3x^3 - x^2}$

15. $f(x) = 5 - \dfrac{3}{x}$

16. $f(x) = 4 + \dfrac{2}{x}$

17. $f(x) = \dfrac{8x^4 - 5x^2}{2x^3 + x^2}$

18. $f(x) = \dfrac{6x^3 + 4x}{3x^2 - x}$

19. $f(x) = \dfrac{6x^4 + 4x^2 - 7}{2x^5 - x + 3}$

20. $f(x) = \dfrac{4x^3 - 3x + 2}{x^3 + 2x - 4}$

21. $f(x) = \dfrac{2x^3 - 4x + 1}{4x^3 + 2x - 3}$

22. $f(x) = \dfrac{5x^4 - 2x^3 + x}{x^5 - x^3 + 8}$

31. $f(x) = \dfrac{3x - 1}{x}$

32. $f(x) = \dfrac{2x + 1}{x}$

33. $f(x) = x + \dfrac{2}{x}$

34. $f(x) = x + \dfrac{9}{x}$

35. $f(x) = \dfrac{-1}{x^2}$

36. $f(x) = \dfrac{2}{x^2}$

37. $f(x) = \dfrac{x}{x + 2}$

38. $f(x) = \dfrac{x}{x - 3}$

39. $f(x) = \dfrac{-1}{x^2 + 2}$

40. $f(x) = \dfrac{1}{x^2 + 3}$

41. $f(x) = \dfrac{x + 3}{x^2 - 9}$ (*Hint:* Simplify.)

42. $f(x) = \dfrac{x - 1}{x^2 - 1}$

43. $f(x) = \dfrac{x - 1}{x + 2}$

44. $f(x) = \dfrac{x - 2}{x + 1}$

45. $f(x) = \dfrac{x^2 - 4}{x + 3}$

46. $f(x) = \dfrac{x^2 - 9}{x + 1}$

47. $f(x) = \dfrac{x + 1}{x^2 - 2x - 3}$

48. $f(x) = \dfrac{x - 3}{x^2 + 2x - 15}$

49. $f(x) = \dfrac{2x^2}{x^2 - 16}$

50. $f(x) = \dfrac{x^2 + x - 2}{2x^2 - 2}$

51. $f(x) = \dfrac{1}{x^2 - 1}$

52. $f(x) = \dfrac{10}{x^2 + 4}$

53. $f(x) = \dfrac{x^2 + 1}{x}$

54. $f(x) = \dfrac{x^3}{x^2 - 1}$

55. $f(x) = \dfrac{x^2 - 9}{x - 3}$

56. $f(x) = \dfrac{x^2 - 16}{x + 4}$